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Conformal Invariance and Surface Defects in the Two-Dimensional Ising Model. Exact Results

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The surface critical behavior of the two-dimensional Ising model with homogeneous perturbations in the surface interactions is studied on the one-dimensional quantum version. A transfer-matrix method leads to an eigenvalue equation for the excitation energies. The spectrum at the bulk critical point is obtained using an L^{-1} expansion, where L is the length of the Ising chain. It exhibits the towerlike structure which is characteristic of conformal models in the case of irrelevant surface perturbations ($h_s/J_s \neq 0$) as well as for the relevant perturbation $h_s = 0$ for which the surface is ordered at the bulk critical point leading to an extraordinary surface transition. The exponents are deduced from the gap amplitudes and confirmed by exact finite-size scaling calculations. Both cases are finally related through a duality transformation.

KEY WORDS: Ising model; surface defects; surface critical behavior; conformal invariance; finite-size scaling.

1. INTRODUCTION

In a recent work[(1)] (hereafter referred to as I), we presented a numerical study of the critical behavior near a surface defect in the two-dimensional Ising model. The thermal and magnetic surface exponents x_e^s and x_m^s were obtained using a finite-size scaling analysis[(2)] of the surface energy and magnetization or, assuming conformal invariance to hold in this situation, deduced from the appropriate gap amplitudes.[(3)] We worked on the 1D quantum version of the model with chains of up to $L = 2000$ spins. Since then we realized that the eigenvalue equation for the fermion excitations which was numerically studied in I can be exactly solved for the low-lying

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excitations using an L^{-1} expansion. Here we present this exact analysis together with complementary results concerning the case where the surface transverse field h_s vanishes. When $h_s = 0$ the perturbation is relevant and one expects an extraordinary transition[4–6] with $x_e^{\rm ext} = x_m^{\rm ext} = 2$. This is actually what we observe on the second or more distant surface sites, while another critical behavior is obtained on the first site.

In Section 2, we recall the main steps leading to the eigenvalue equation for the excitations. The spectrum and finite-size scaling results are presented in Section 3 for the case of irrelevant surface perturbations and in Section 4 for $h_s = 0$. A duality transformation allowing us to relate both situations is discussed in Section 5.

2. EIGENVALUE EQUATION

Let us consider the 1D quantum version of the 2D Ising model[7] on a chain with L spins and free boundary conditions:

$$\begin{aligned} \mathscr{H} = &-\sum_{i=2}^{L-1} \sigma_z(i) - \sum_{i=2}^{L-2} \sigma_x(i)\,\sigma_x(i+1) - h_s[\sigma_z(1) + \sigma_z(L)] \\ &- J_s[\sigma_x(1)\,\sigma_x(2) + \sigma_x(L-1)\,\sigma_x(L)] \end{aligned} \tag{2.1}$$

where the σ are Pauli spin operators. The system is taken at its critical point $h/J = 1$ in the bulk and perturbed on its surfaces, where the coupling J_s and the transverse field h_s deviate from their critical values. $\mathscr{H}$ commutes with the parity operator $P = \prod_{i=1}^{L} \sigma_z(i)$, allowing us to classify the eigenstates according to their parity in the even ($P = +1$) or odd ($P = -1$) sectors.

A Jordan–Wigner transformation[8] of the spin operators leads to the fermion Hamiltonian

$$\begin{aligned} \mathscr{H} = &-\sum_{i=2}^{L-1} [2c^+(i)\,c(i) - 1] - \sum_{i=2}^{L-2} [c^+(i) - c(i)][c^+(i+1) + c(i+1)] \\ &- 2h_s[c^+(1)\,c(1) + c^+(L)\,c(L) - 1] \\ &- J_s\{[c^+(1) - c(1)][c^+(2) + c(2)] \\ &+ [c^+(L-1) - c(L-1)][c^+(L) + c(L)]\} \end{aligned} \tag{2.2}$$

This quadratic form in the fermion operators may be diagonalized using standard methods,[9] giving

$$\mathscr{H} = E_0 + \sum_k \Lambda_k \eta_k^+ \eta_k \tag{2.3}$$

where the excitations Λ_k are obtained as solutions of the eigenvalue equation

$$\bar{\bar{\Lambda}}\bar{\phi}_k = \Lambda_k^2 \bar{\phi}_k \tag{2.4}$$

with

$$\bar{\bar{\Lambda}} = (\bar{\bar{A}} - \bar{\bar{B}})(\bar{\bar{A}} + \bar{\bar{B}}) \tag{2.5}$$

where $\bar{\bar{A}}$ and $\bar{\bar{B}}$ are the following tridiagonal matrices:

$$\bar{\bar{A}} = -\begin{pmatrix} 2h_s & J_s & & & & & \\ J_s & 2 & 1 & & 0 & & \\ & 1 & 2 & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & 2 & 1 & \\ & 0 & & & 1 & 2 & J_s \\ & & & & & J_s & 2h_s \end{pmatrix} \tag{2.6}$$

$$\bar{\bar{B}} = \begin{pmatrix} 0 & -J_s & & & & & \\ J_s & 0 & -1 & & 0 & & \\ & 1 & 0 & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & 0 & -1 & \\ & 0 & & & 1 & 0 & -J_s \\ & & & & & J_s & 0 \end{pmatrix} \tag{2.7}$$

such that

$$(\bar{\bar{A}} + \bar{\bar{B}})\,\bar{\phi}_k = \Lambda_k \bar{\psi}_k \tag{2.8}$$

$$(\bar{\bar{A}} - \bar{\bar{B}})\,\bar{\psi}_k = \Lambda_k \Phi_k \tag{2.9}$$

and the normalized eigenvectors $\bar{\phi}$ and $\bar{\psi}$ are linear combinations of the coefficients of the canonical transformation leading to Eq. (2.3) (see I). Introducing two-components column vectors, we replace the eigenvalue equation (2.4) by the recursion[10]

$$\begin{vmatrix} \varphi(i) \\ \varphi(i+1) \end{vmatrix} = \bar{\bar{T}}_i \begin{vmatrix} \varphi(i-1) \\ \varphi(i) \end{vmatrix} = \begin{pmatrix} 0 & 1 \\ s_i & t_i \end{pmatrix} \begin{vmatrix} \varphi(i-1) \\ \varphi(i) \end{vmatrix} \tag{2.10}$$

with the boundary conditions $\varphi(0) = \varphi(L+1) = 0$ and $\varphi(1) = 1$. The $\bar{\varphi}$ in

Eq. (2.10) coincide with the $\bar{\phi}$ when normalized. The 2×2 transfer matrix $\bar{\bar{T}}_i$ has the following matrix elements:

$$t_i = t = \frac{\Lambda_k^2}{4} - 2, \qquad s_i = s = -1 \quad (i = 3 \text{ to } L-2) \tag{2.11a}$$

$$t_1 = \frac{\Lambda_k^2}{4h_s J_s} - \frac{h_s}{J_s}, \qquad s_1 = 0 \tag{2.11b}$$

$$t_2 = \frac{\Lambda_k^2}{4} - J_s^2 - 1, \qquad s_2 = -h_s J_s \tag{2.11c}$$

$$t_{L-1} = \frac{\Lambda_k^2}{4J_s} - \frac{2}{J_s}, \qquad s_{L-1} = -\frac{1}{J_s} \tag{2.11d}$$

$$t_L = \frac{\Lambda_k^2}{4J_s} - \frac{h_s^2}{J_s} - J_s, \qquad s_L = -1 \tag{2.11e}$$

and in order to satisfy the boundary conditions the eigenvalues must be such that

$$\begin{vmatrix} \varphi(L) \\ 0 \end{vmatrix} = \bar{\bar{T}}_L \bar{\bar{T}}_{L-1} \bar{\bar{T}}^{L-4} \bar{\bar{T}}_2 \bar{\bar{T}}_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{2.12}$$

where $\bar{\bar{T}} = \bar{\bar{T}}_i$ $(i = 3$ to $L-2)$.

The low-lying excitations $(\Lambda_k < 4)$ correspond to propagating modes. Then $\bar{\bar{T}}$ has two complex conjugate eigenvalues

$$\lambda_\pm = \theta \pm i(1-\theta^2)^{1/2} = e^{\pm ip} \tag{2.13}$$

with $\theta = t/2$. Using the corresponding eigenvectors $\bar{v}_\pm = \left|{}^{\,1}_{\lambda_\pm}\right|$, one may write

$$\bar{\bar{T}}_2 \bar{\bar{T}}_1 \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \alpha \bar{v}_+ + \beta \bar{v}_- \tag{2.14}$$

and identifying the lower components on both sides of Eq. (2.12), one is led to the following equation for the propagating modes[2]:

$$\begin{aligned} &(t_L t_{L-1} - 1)[(\alpha+\beta)\cos(L-3)p + i(\alpha-\beta)\sin(L-3)p] \\ &\quad - (t_L/J_s)[(\alpha+\beta)\cos(L-4)p + i(\alpha-\beta)\sin(L-4)p] = 0 \end{aligned} \tag{2.15}$$

with

$$\alpha + \beta = t_1 \tag{2.16a}$$

$$i(\alpha - \beta) = \frac{t_1 t_2 - h_s J_s - t_1 \theta}{(1-\theta^2)^{1/2}} \tag{2.16b}$$

[2] The first factor in Eq. (2.15) was incorrectly transcribed in Eqs. (4.14) and (4.16) of I.

3. EXCITATION SPECTRUM, SURFACE ENERGY, AND SURFACE MAGNETIZATION WHEN $h_s/J_s \neq 0$

At the bulk critical point, when the size L of the system grows to infinity, the low-lying excitations are known to vanish like[11]

$$\Lambda_k = \frac{2a_k}{L}\left(1 + \frac{b_k}{L}\right) + O(L^{-3}) \tag{3.1}$$

An expansion of Eq. (2.13) in powers of L^{-1} gives

$$\lambda_\pm = -1 \pm i\frac{a_k}{L} + \left(\frac{a_k}{2} \pm ib_k\right)\frac{a_k}{L^2} + O(L^{-3}) \tag{3.2a}$$

$$p = \pi - \left(1 + \frac{b_k}{L}\right)\frac{a_k}{L} + O(L^{-3}) \tag{3.2b}$$

whereas Eqs. (2.16a) and (2.16b) lead to

$$\alpha + \beta = -\frac{h_s}{J_s} + O(L^{-2}) \tag{3.3a}$$

$$i(\alpha - \beta) = -\left(\frac{J_s}{h_s} + \frac{h_s}{2J_s}\right)\frac{a_k}{L} + O(L^{-2}) \tag{3.3b}$$

From Eqs. (2.11d) and (2.11e), one deduces

$$t_L t_{L-1} - 1 = 1 + 2\left(\frac{h_s}{J_s}\right)^2 + O(L^{-2}) \tag{3.4a}$$

$$-\frac{t_L}{J_s} = 1 + \left(\frac{h_s}{J_s}\right)^2 + O(L^{-2}) \tag{3.4b}$$

and after some algebra, Eq. (2.15) may by rewritten as

$$\left(\frac{h_s}{J_s}\right)^3 \cos a_k + \left[\left(\frac{3}{2} - b_k\right)\left(\frac{h_s}{J_s}\right)^3 - 2\frac{h_s}{J_s}\right]\frac{a_k}{L}\sin a_k = O(L^{-2}) \tag{3.5}$$

When $h_s/J_s \neq 0$ the leading term gives $\cos a_k = 0$ and

$$a_k = (2k-1)\frac{\pi}{2} \tag{3.6}$$

Up to $O(L^{-1})$ one gets the coefficient of the first correction to scaling:

$$b_k = \frac{3}{2} - 2\left(\frac{J_s}{h_s}\right)^2 \tag{3.7}$$

so that

$$\Lambda_k=(2k-1)\frac{\pi}{L}+\left[\frac{3}{2}-2\left(\frac{J_s}{h_s}\right)^2\right](2k-1)\frac{\pi}{L^2}+O(L^{-3}) \qquad (k=1,2,...) \tag{3.8}$$

As a consequence of the irrelevance of the surface perturbation when $h_s/J_s \neq 0$, the excitation spectrum, up to terms of order L^{-1}, is the same as for the unperturbed surface.

The gap of the sound velocity v_s is given, for free boundaries, by[12]

$$G_{v_s}=v_s\frac{\pi}{L}+O(L^{-2})=\Lambda_{k+1}-\Lambda_k \tag{3.9}$$

and $v_s=2$, as expected for the Ising model in a transverse field.[13] The anomalous dimension of the surface energy operator may be deduced from the first gap in the even sector

$$G_e=v_s\frac{\pi}{L}x_e^s+O(L^{-2})=\Lambda_1+\Lambda_2 \tag{3.10}$$

and the surface magnetization exponent from the first gap in the odd sector:

$$G_m=v_s\frac{\pi}{L}x_m^s+O(L^{-2})=\Lambda_1 \tag{3.11}$$

leading to the exponents of the ordinary surface transition[14]:

$$x_e^s=2, \qquad x_m^s=1/2 \tag{3.12}$$

One gets two conformal towers, one for each parity sector (Fig. 1):

$$E_{n+1}^{\text{even}}-E_0=\Lambda_1+\Lambda_{n+2}=v_s\frac{\pi}{L}(x_e^s+n)+O(L^{-2}) \tag{3.13a}$$

$$E_{n+1}^{\text{odd}}-E_0=\Lambda_{n+1}=v_s\frac{\pi}{L}(x_m^s+n)+O(L^{-2}) \tag{3.13b}$$

where n is a nonnegative integer and we have given one of the combinations of excitations leading to the excited levels. Since the ground state is even, with an even (odd) number of excitations, one reaches an even (odd) excited state.

Let us now turn to the finite-size scaling analysis. The energy and the

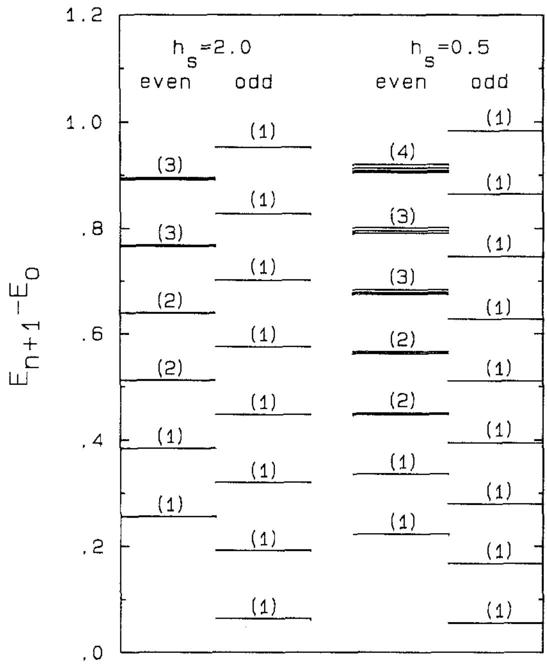

Fig. 1. Lower levels of the conformal towers for the ordinary surface transition with $L=50$, a surface coupling $J_s=1$, and a surface transverse field $h_s=2$ or 0.5. The level degeneracy is given in parentheses. An $O(L^{-2})$ perturbation-dependent level shift is observed.

magnetization may be studied on the surface site using the following matrix elements (see I):

$$e_s^z(1)=\langle\varepsilon|\,\sigma_z(1)\,|0\rangle\sim L^{-x_e^s} \tag{3.14a}$$

$$m_s(1)=\langle\sigma|\,\sigma_x(1)\,|0\rangle\sim L^{-x_m^s} \tag{3.14b}$$

between the ground state $|0\rangle$ and the first excited states, $|\varepsilon\rangle=\eta_1^+\eta_2^+\,|0\rangle$ in the even sector and $|\sigma\rangle=\eta_1^+\,|0\rangle$ in the odd sector. Working with the canonical fermion representation, one easily gets

$$e_s^z(i)=\psi_1(i)\,\phi_2(i)-\psi_2(i)\,\phi_1(i) \tag{3.15a}$$

$$m_s(1)=\phi_1(1) \tag{3.15b}$$

The eigenvectors may be obtained through the transfer matrix method:

$$\begin{vmatrix}\varphi(n-1)\\ \varphi(n)\end{vmatrix}=\bar{\bar{T}}^{n-3}\bar{\bar{T}}_2\bar{\bar{T}}_1\begin{vmatrix}0\\ 1\end{vmatrix}\qquad(n=3\text{ to }L-1) \tag{3.16}$$

giving

$$\varphi(n) = (\alpha + \beta) \cos(n-2) p + i(\alpha - \beta) \sin(n-2) p \tag{3.17}$$

with

$$p = \pi - (2k-1) \frac{\pi}{2L} + O(L^{-2}) \tag{3.18}$$

After some algebra, one gets

$$\sum_{n=1}^{L} \varphi^2(n) = \frac{1}{2} \left(\frac{h_s}{J_s}\right)^2 L + O(1) \tag{3.19}$$

so that the first component of the normalized eigenvectors

$$\phi_k(1) = \sqrt{2} \left(\frac{J_s}{h_s}\right) L^{-1/2} + O(L^{-3/2}) \tag{3.20}$$

does not depend on k to this order. The ψ_k may be deduced from Eq. (2.8), leading to

$$\psi_k(1) = -\frac{(2k-1)\,\pi J_s}{\sqrt{2}\,h_s^2} L^{-3/2} + O(L^{-5/2}) \tag{3.21}$$

Finally, using Eqs. (3.15a) and (3.15b), we get

$$e_s^z(1) = \frac{2\pi}{h_s} \left(\frac{J_s}{h_s}\right)^2 L^{-2} + O(L^{-3}) \tag{3.22}$$

and

$$m_s(1) = \sqrt{2} \left(\frac{J_s}{h_s}\right) L^{-1/2} + O(L^{-3/2}) \tag{3.23}$$

4. EXCITATION SPECTRUM, SURFACE ENERGY, AND SURFACE MAGNETIZATION WHEN $h_s = 0$

When $h_s = 0$ the squares of the excitation energies are the eigenvalues of the following $L \times L$ matrix:

$$\bar{\bar{\Lambda}} = 4 \begin{pmatrix} 0 & 0 & & & & \\ 0 & 1+J_s^2 & 1 & & 0 & \\ & 1 & 2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & 2 & J_s \\ & & & & J_s & J_s^2 \end{pmatrix} \tag{4.1}$$

As a consequence, the lowest one is $\Lambda_0^2 = 0$ and the remaining $L-1$ are eigenvalues of the $(L-1)\times(L-1)$ tridiagonal submatrix to which the same transfer matrix technique may be applied. $\bar{\bar{T}}$, $\bar{\bar{T}}_{L-1}$, t_2, and s_L remain unchanged, but now

$$s_2 = 0, \qquad t_L = \frac{\Lambda_k^2}{4J_s} - J_s \tag{4.2}$$

and the recursion for the $\bar{\varphi}$ reads

$$\begin{vmatrix} \varphi(L) \\ 0 \end{vmatrix} = \bar{\bar{T}}_L \bar{\bar{T}}_{L-1} \bar{\bar{T}}^{L-4} \bar{\bar{T}}_2 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{4.3}$$

With

$$\bar{\bar{T}}_2 \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \gamma \bar{v}_+ + \delta \bar{v}_- \tag{4.4}$$

the excitations corresponding to propagating modes are still given by Eq. (2.15) provided α and β are changed into γ and δ. Looking for the excitations in the form given in Eq. (3.1), we get

$$\gamma + \delta = 1 \tag{4.5a}$$

$$i(\gamma - \delta) = \frac{t_2 - \theta}{(1-\theta^2)^{1/2}} = -\frac{J_s^2}{a_k}(L - b_k) + O(L^{-1}) \tag{4.5b}$$

$$t_L t_{L-1} - 1 = 1 + O(L^{-2}) \tag{4.5c}$$

$$-\frac{t_L}{J_s} = 1 + O(L^{-2}) \tag{4.5d}$$

and Eq. (2.15) leads to

$$-J_s^2 \cos a_k = O(L^{-1}) \tag{4.6}$$

to the leading order in L^{-1}, so that $a_k = (2k-1)\,\pi/2$ and

$$\Lambda_k = (2k-1)\frac{\pi}{L} + O(L^{-2}) \qquad (k = 1, 2, \ldots) \tag{4.7}$$

The sound velocity v_s is still given by Eq. (3.3), but due to the presence of the vanishing excitation $\Lambda_0 = 0$, we now get

$$G_e = v_s \frac{\pi}{L} x_e^s + O(L^{-2}) = \Lambda_0 + \Lambda_1 \tag{4.8}$$

and

$$G_m = v_s \frac{\pi}{L} x_m^s + O(L^{-2}) = \Lambda_0 \tag{4.9}$$

so that

$$x_e^s = \tfrac{1}{2}, \qquad x_m^s = 0 \tag{4.10}$$

The vanishing magnetic exponent signals the occurrence of surface long-range order at the bulk critical point when $h_s = 0$. In the 2D classical model $h_s = 0$ corresponds to an infinite surface coupling in the temporal direction; the surface spins are then frozen in a ferromagnetic configuration and one should observe a behavior characteristic of the surface extraordinary transition on nearby sites. To get these exponents, one has to look at higher degenerate excited states with

$$G_e^{\text{ext}} = v_s \frac{\pi}{L} x_e^{\text{ext}} + O(L^{-2}) = \Lambda_1 + \Lambda_2 \tag{4.11a}$$

$$G_m^{\text{ext}} = v_s \frac{\pi}{L} x_m^{\text{ext}} + O(L^{-2}) = \Lambda_0 + \Lambda_1 + \Lambda_2 \tag{4.11b}$$

leading to

$$x_e^{\text{ext}} = 2, \qquad x_m^{\text{ext}} = 2 \tag{4.12}$$

as required for the extraordinary transition.[6] Surface long-range order with $\Lambda_0 = 0$ leads to an even–odd degeneracy (Fig. 2) and we get the following conformal towers:

$$E_{n+1}^{\text{odd(even)}} - E_0 = \Lambda_{n+1}(+\Lambda_0) = v_s \frac{\pi}{L}(x_e^s + n) + O(L^{-2}) \tag{4.13}$$

$$E_{n+1}^{\text{ext even(odd)}} - E_0 = \Lambda_1 + \Lambda_{n+2}(+\Lambda_0) = v_s \frac{\pi}{L}(x_{e(m)}^{\text{ext}} + n) + O(L^{-2}) \tag{4.14}$$

the last one corresponding to the extraordinary transition. The state leading to $x_m^s = 0$, which is degenerate with the ground state, does not belong to one of the conformal towers. This may be linked to the fact that $x_m^s = 0$, associated with a regular constant contribution to the surface magnetization, is not really an anomalous dimension.

These results are confirmed by a finite-size scaling analysis of the surface critical behavior. In order to do this, we need the components of

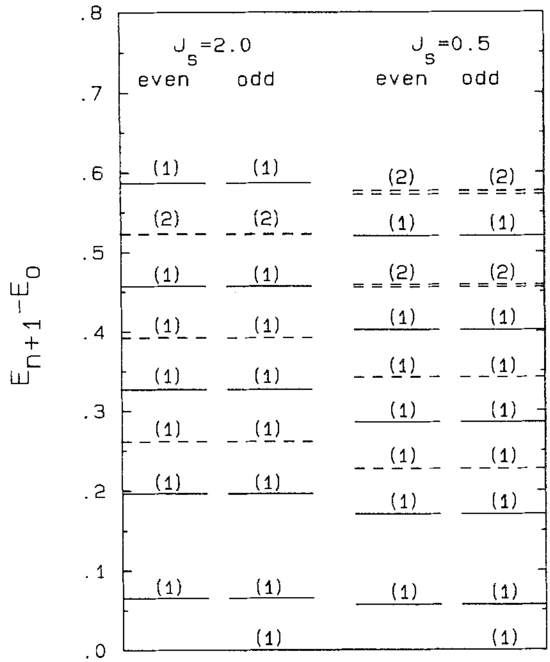

Fig. 2. Lower levels of the conformal towers with $L=50$, a vanishing surface transverse field, and surface coupling $J_s=2$ or 0.5. The levels of the towers associated with the extraordinary transition are drawn in dashed lines. The level degeneracy is indicated in parentheses. As in Fig. 1, a level shift due to the J_s dependence of the $O(L^{-2})$ corrections is apparent.

the normalized eigenvectors $\bar{\phi}$ and $\bar{\psi}$ on the first two sites. From Eq. (2.8) with $h_s=0$, one deduces

$$\Lambda_k \psi_k(1) = -2J_s \phi_k(2) \qquad (k=0, 1, 2,...) \tag{4.15}$$

so that $\phi_0(2)$ vanishes with Λ_0. Equations (2.4) and (4.1) provide a recursion for the $\phi_0(n)$ and one may verify that they all vanish when $n>1$. Through normation one gets

$$\phi_0(n) = \delta_{n,1} \tag{4.16}$$

In the same way, Eq. (2.9) gives a recursion for the $\psi_0(n)$ and

$$\psi_0(n) = \delta_{n,L} \tag{4.17}$$

Using Eqs. (2.4) and (4.1), one obtains

$$\Lambda_k^2 \phi_k(1) = 0 \tag{4.18}$$

so that

$$\phi_k(1) = 0 \qquad (k = 1, 2, ...) \tag{4.19}$$

With the same procedure as in Section 3 applied to the $(L-1)\times(L-1)$ submatrix, one gets a recursion relation for the $\varphi(n)$ corresponding to the excited states $(k>0)$ with $\varphi(2)=1$ and $\varphi(1)=0$ from Eq. (4.19). This is Eq. (3.17) with α and β changed into γ and δ. After some calculations one gets

$$\sum_{n=1}^{L} \varphi^2(n) = \frac{2J_s^4}{(2k-1)^2\,\pi^2} L^3 + O(L^2) \tag{4.20}$$

and, through normation,

$$\phi_k(2) = \frac{(2k-1)\,\pi}{\sqrt{2}\,J_s^2} L^{-3/2} + O(L^{-5/2}) \qquad (k=1, 2, ...) \tag{4.21}$$

Finally, Eq. (4.15) gives

$$\psi_k(1) = -\frac{\sqrt{2}}{J_s} L^{-1/2} + O(L^{-3/2}) \qquad (k=1, 2, ...) \tag{4.22}$$

Equation (4.16) leads to

$$m_s(1) = \phi_0(1) = 1 \tag{4.23}$$

and Eqs. (4.16), (4.17), and (4.22) to

$$e_s^z(1) = \psi_0(1)\,\phi_1(1) - \psi_1(1)\,\phi_0(1) = \frac{\sqrt{2}}{J_s} L^{-1/2} + O(L^{-3/2}) \tag{4.24}$$

confirming the values $x_m^s = 0$, $x_e^s = 1/2$.

The exponents of the extraordinary transition may be deduced from[3] $\langle \varepsilon' |\, \sigma_x(1)\,\sigma_x(2)\, |0\rangle$ with $|\varepsilon'\rangle = \eta_1^+ \eta_2^+\, |0\rangle$ for the surface energy, so that

$$\begin{aligned} e_s^{xx}(1, 2) &= \psi_1(1)\,\phi_2(2) - \psi_2(1)\,\phi_1(2) \\ &= -\frac{2\pi}{J_s^3} L^{-2} + O(L^{-3}) \end{aligned} \tag{4.25}$$

[3] One should notice that the excitation spectrum begins now with $k = 0$.

and from $\langle\sigma'|\,\sigma_x(2)\,|0\rangle$ with $|\sigma'\rangle = \eta_0^+\eta_1^+\eta_2^+\,|0\rangle$ for the surface magnetization on the second site

$$\begin{aligned} m_s(2) &= 2\phi_1(2)\,\phi_2(2)\left(\frac{1}{\Lambda_2}-\frac{1}{\Lambda_1}\right) \\ &= -\frac{2\pi}{J_s^4}L^{-2}+O(L^{-3}) \end{aligned} \tag{4.26}$$

in agreement with $x_m^{\text{ext}} = x_e^{\text{ext}} = 2$.

5. DUALITY TRANSFORMATION AND EXTRAORDINARY TRANSITION WITH $h_s = 0$

With a vanishing surface transverse field the spin Hamiltonian reads

$$\begin{aligned} \mathscr{H} = &-\sum_{i=2}^{L-1}\sigma_z(i) - \sum_{i=2}^{L-2}\sigma_x(i)\,\sigma_x(i+1) \\ &- J_s[\sigma_x(1)\,\sigma_x(2)+\sigma_x(L-1)\,\sigma_x(L)] \end{aligned} \tag{5.1}$$

Defining the dual spins as

$$\mu_z(i) = \sigma_x(i)\,\sigma_x(i+1) \qquad (1 \leqslant i < L) \tag{5.2a}$$

$$\mu_z(L) = \sigma_x(L) \tag{5.2b}$$

$$\mu_x(i) = \prod_{j=1}^{i}\sigma_z(j) \qquad (1 \leqslant i \leqslant L) \tag{5.2c}$$

where boundary effects have been taken into account in Eq. (5.2b), we get the following dual Hamiltonian:

$$\mathscr{H}_D = -\sum_{i=2}^{L-1}\mu_x(i-1)\,\mu_x(i) - \sum_{i=2}^{L-2}\mu_z(i) - J_s[\mu_z(1)+\mu_z(L)] \tag{5.3}$$

corresponding to a chain containing $L-1$ spins with free ends and a free spin $\mu(L)$ in the transverse field J_s. The perturbation introduced on the first spin of the chain by J_s is irrelevant as long as J_s remains finite and non-vanishing. As already mentioned in I, $\sigma_z(1)$ transforms into $\mu_x(1)$ and the surface energy operator scales like the surface magnetization on the dual with $x_m^s = 1/2$. On the other hand, $\sigma_x(1)$ transforms into the dual parity operator, which is size independent.

The duality transformation allows us to get the exponents of the extraordinary transition.[6,15] The surface energy operators $\sigma_x(1)\,\sigma_x(2)$ and $\sigma_z(2)$ transform into the surface energy operators $\mu_z(1)$ and $\mu_x(1)\,\mu_x(2)$ on the dual chain, so that $x_e^{\text{ext}} = x_e^s = 2$. The surface magnetization $\sigma_x(L-1)$ gives the dual operator $\mu_z(L-1)\,\mu_z(L)$, but $\mu_z(L)$ on the free spin gives back $\sigma_x(L)$ with vanishing dimension, so that the surface magnetization on the second site scales like the surface energy on the dual chain with $x_m^{\text{ext}} = x_e^s = 2$.

6. CONCLUSION

Using a transfer-matrix technique together with an L^{-1} expansion for the excitation spectrum, we have shown that the towerlike structure which is characteristic for conformal models is conserved when perturbations are introduced on the surface of the 2D Ising model, working on the 1D quantum version. This is verified both in the case of irrelevant perturbations ($h_s/J_s \neq 0$), where the gap amplitudes give the ordinary surface exponents, and for a relevant perturbation ($h_s = 0$), for which the surface is ordered and an extraordinary surface transition is obtained. Then the surface energy on the first site has a scaling dimension $x_e^s = 1/2$, which is characteristic of the quantum Hamiltonian version of the problem. In the 2D classical system, $h_s = 0$ corresponds to an infinite coupling along the surface row, so that energy fluctuations are excluded. Since the first excitation has a vanishing energy, one gets a pair of degenerate conformal towers associated with the extraordinary transition. As a consequence the thermal and magnetic exponents, related to the first gap, are the same.

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